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A characterization of the essential pseudospectra on a Banach space

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Abstract In this paper, we introduce and study the essential pseudospectra of closed, densely defined linear operators in the Banach space. We start by giving the definition and we investigate the characterization, the stability and some properties of these essential pseudospectra.

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الملخص

في هذه الورقة نقدّم وندرس أشباه الأطياف الجوهرية لمؤثرات خطية ومغلقة ومعرفة بكثافة على فضاءات بناخ. نبدأ بإعطاء التعريف ثم نبحث التمييز، والاستقرار وبعض خصائص أشباه الأطياف الجوهرية هذه.

1 Introduction

Let X and Y be two Banach spaces. We denote by $\mathcal{L}(X, Y)$ $\big($resp., $\mathcal{C}(X, Y)\big)$ the set of all bounded (resp., closed, densely defined) linear operators from X into Y and we denote by $\mathcal{K}(X, Y)$ the subspace of compact operators from X into Y. An operator $T \in \mathcal{L}(X, Y)$ is said to be weakly compact if $T(B)$ is relatively weakly compact in Y for every bounded subset $B \subset X$. The family of weakly compact operators from X into Y is denoted by $\mathcal{W}(X, Y)$. For $T \in \mathcal{C}(X, Y)$, we denote by $\sigma(T)$, $\rho(T)$, $N(T)$ and $R(T)$, respectively, the spectrum, the resolvent set, the null space and the range of T. The nullity, $\alpha(T)$, of T is defined as the dimension of $N(T)$ and the deficiency, $\beta(T)$, of T is defined as the codimension of $R(T)$ in Y. The set of upper semi-Fredholm operators from X into Y is defined by

$$\Phi_+(X, Y) := \{T \in \mathcal{C}(X, Y) \text{ such that } \alpha(T) < \infty \text{ and } R(T) \text{ is closed in } Y\},$$

the set of lower semi-Fredholm operators from X into Y is defined by

$$\Phi_-(X, Y) := \big\{T \in \mathcal{C}(X, Y) \text{ such that } \beta(T) < \infty \text{ and } R(T) \text{ is closed in } Y\big\}.$$

$\Phi_\pm(X, Y) := \Phi_+(X, Y) \cup \Phi_-(X, Y)$ denotes the set of semi-Fredholm operators from X into Y and $\Phi(X, Y) := \Phi_+(X, Y) \cap \Phi_-(X, Y)$ denotes the set of Fredholm operators on X into Y, the set of bounded Fredholm operators from X into Y is defined by

$$\Phi^b(X, Y) = \Phi(X, Y) \cap \mathcal{L}(X, Y).$$

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If $X = Y$, the sets $\mathcal{L}(X, Y)$, $\mathcal{C}(X, Y)$, $\Phi(X, Y)$, $\Phi_+(X, Y)$, $\Phi_-(X, Y)$ and $\Phi^b(X, Y)$are replaced, by $\mathcal{L}(X)$, $\mathcal{C}(X)$, $\Phi(X)$, $\Phi_+(X)$, $\Phi_-(X)$ and $\Phi^b(X)$, respectively. For $T \in \Phi_\pm(X)$, the number $i(T) = \alpha(T) - \beta(T)$ is called the index of T. It is clear that if $T \in \Phi(X)$ then $i(T) < \infty$. If $T \in \Phi_+(X) \setminus \Phi(X)$ then $i(T) = -\infty$ and if $T \in \Phi_-(X) \setminus \Phi(X)$ then $i(T) = +\infty$. An operator $F \in \mathcal{L}(X, Y)$ is called a Fredholm perturbation, if $T + F \in \Phi(X, Y)$ whenever $T \in \Phi(X, Y)$. We denote by $\mathcal{F}(X)$ the set of Fredholm perturbations. A Banach space X is said to have the Dunford-Pettis property (for short property DP) if for each Banach space Y every weakly compact operator $T : X \longrightarrow Y$ takes weakly compact sets in X into norm compact sets of Y. For example it is well known that any L_1-space has the property DP. An operator $S \in \mathcal{L}(X)$ is called strictly singular if, for every infinite-dimensional subspace M of X, the restriction of S to M is not a homeomorphism. Let T be a closed linear operator on a Banach space X. For $x \in \mathcal{D}(T)$ the graph norm of x is defined by

$$\|x\|_T := \|x\| + \|Tx\|.$$

It follows from the closedness of T that $\mathcal{D}(T)$ endowed with the norm $\|.\|_T$ is a Banach space. Let X_T denote $(\mathcal{D}(T), \|.\|_T)$. In this new space the operator T satisfies $\|Tx\| \leq \|x\|_T$ and consequently T is a bounded operator from X_T into X. If $\hat{T}$ denotes the restriction of T to $\mathcal{D}(T)$, we observe that $\alpha(\hat{T}) = \alpha(T)$ and $\beta(\hat{T}) = \beta(T)$.

Definition 1.1 [18] Let A be a linear operator from X to Y. A linear operator B from X to Y is called A-compact if $\mathcal{D}(A) \subset \mathcal{D}(B)$ and whenever a sequence $\{x_k\}$ of elements of $\mathcal{D}(A)$ satisfies

$$\|x_k\| + \|Ax_k\| \leq c, \quad k = 1, 2 \ldots,$$

then $\{Bx_k\}$ has a subsequence convergent in Y.

There are several and in general non-equivalent definitions of the essential spectrum of a linear operator on a Banach space. For a self-adjoint operator in a Hilbert space, there seems to be only one reasonable way to define the essential spectrum: the set of all points of the spectrum that are not isolated eigenvalues of finite algebraic multiplicity. In this paper, we are interested by the following essential spectra:

$$\sigma_w(T) := \mathbb{C} \setminus \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \in \Phi(X) \text{ and } i(\lambda - T) = 0\},$$
$$\sigma_b(T) := \sigma(T) \setminus \sigma_d(T),$$

where $\sigma_d(T)$ is the set of isolated points λ of the spectrum such that the corresponding Riesz projectors P_λ is finite rank operator with kernel denote by K_λ. $\sigma_w(.)$ is the Wely spectrum (see for example [6,11,17,18]) and $\sigma_b(.)$ is the Browder spectrum [2,14,16].

It is well established that the spectrum of a self-adjoint operator is of crucial importance in understanding its action in various applied contexts. For highly non-self-adjoint operators, on the other hand, there is increasing evidence that the spectrum is often not very helpful, and that the pseudospectrum is of more importance. The definition of pseudospectrum of closed densely linear operator T for every $\varepsilon > 0$ is given by:

$$\sigma_\varepsilon(T) := \sigma(T) \cup \left\{\lambda \in \mathbb{C} \text{ such that } \|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon}\right\}.$$

By convention we write $\|(\lambda - T)^{-1}\| = \infty$ if $(\lambda - T)^{-1}$ is unbounded or nonexistent, i.e., if λ is in the spectrum $\sigma(T)$. This means that the pseudospectrum can be introduced as a zone of spectral instability. The concept of pseudospectrum was introduced perhaps first time by J. M. Varah [21] and has been subsequently employed by other authors for example, H. Landau [12], L. N. Trefethen [19], D. Hinrichsen et al [7] and E. B. Davies [3]. Especially due to L. N. Trefethen, who developed this idea for matrices and operators, he used this concept to the study of interesting problems in mathematical physics.

In [1] F. Abdmouleh, A. Ammar and A. Jeribi defined the notion of pseudo-Browder essential spectra of densely closed, linear operators in the Banach space by:

$$\sigma_{b,\varepsilon}(T) = \sigma_b(T) \cup \left\{\lambda \in \mathbb{C} \text{ such that } \|R_b(\lambda, T)\| > \frac{1}{\varepsilon}\right\},$$

where $R_b(\lambda, T) = \big((\lambda - T)|_{K_\lambda}\big)^{-1}\big(I - P_\lambda\big) + P_\lambda$ and by convention we write $\|R_b(\lambda, T)\| = \infty$ if $R_b(\lambda, T)$ is unbounded or nonexistent, i.e., if λ is in the spectrum $\sigma_b(T)$.

The notion of Weyl pseudospectrum can be extended by devoting our studies on the essential spectrum and we have by

$$\sigma_{w,\varepsilon}(A) = \bigcap_{K\in\mathcal{K}(X)} \sigma_\varepsilon(A+K).$$

In the following we characterize the Weyl pseudospectra by:
$\lambda \notin \sigma_{w,\varepsilon}(A)$ if and only if, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have

$$A + D - \lambda \in \Phi(X) \text{ and } i(A + D - \lambda) = 0.$$

This is equivalent to say that

$$\sigma_{w,\varepsilon}(A) = \bigcup_{\|D\|<\varepsilon} \sigma_w(A+D).$$

When dealing with Weyl pseudospectra of closed, densely defined linear operators on Banach spaces, one of the main problems consists of studying the invariance of the Weyl pseudospectrum of these operators subjected to various kinds of perturbation. Let $\varepsilon > 0$ and a linear operator A on a Banach space X, the question is what are the conditions that we must impose on the operator $K \in \mathcal{C}(X)$ such that $\sigma_{w,\varepsilon}(A+K) = \sigma_{w,\varepsilon}(A)$. If K is a compact operator on the Banach space X, then the result follows from the definitions of $\sigma_{w,\varepsilon}(.)$ and if K is Fredholm perturbation that $\sigma_{w,\varepsilon}(A+K) = \sigma_{w,\varepsilon}(A)$. In fact, if for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have $A(B+D) \in \mathcal{F}(X)$ and $(B+D)A \in \mathcal{F}(X)$ then

$$\sigma_{w,\varepsilon}(A+B)\backslash\{0\} = \big[\sigma_w(A) \cup \sigma_{w,\varepsilon}(B)\big]\backslash\{0\}.$$

2 Main results

Definition 2.1 Let X be a Banach space, $\varepsilon > 0$ and $A \in \mathcal{C}(X)$. We define the Weyl pseudospectrum of the operator A by

$$\sigma_{w,\varepsilon}(A) = \bigcap_{K\in\mathcal{K}(X)} \sigma_\varepsilon(A+K).$$

Remark 2.2 It follows from Definition 2.1 and the properties of pseudospectra (see, for example [4,20]) that

(i) $\sigma_{w,\varepsilon}(A) \subset \sigma_\varepsilon(A)$.
(ii) $\bigcap_{\varepsilon>0} \sigma_{w,\varepsilon}(A) = \sigma_w(A)$.
(iii) If $\varepsilon_1 < \varepsilon_2$ then $\sigma_w(A) \subset \sigma_{w,\varepsilon_1}(A) \subset \sigma_{w,\varepsilon_2}(A)$.
(iv) $\sigma_{w,\varepsilon}(A+K) = \sigma_{w,\varepsilon}(A)$ for all $K \in \mathcal{K}(X)$.

The following theorem gives a characterization of the Weyl pseudospectrum by means of Fredholm operator.

Theorem 2.3 *Let X be a Banach space, $\varepsilon > 0$ and $A \in \mathcal{C}(X)$. Then $\lambda \notin \sigma_{w,\varepsilon}(A)$ if and only if, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, we have*

$$A + D - \lambda \in \Phi(X) \quad \text{and} \quad i(A + D - \lambda) = 0.$$

Proof Let $\lambda \notin \sigma_{w,\varepsilon}(A)$. Using [4, Theorem 9.2.13] we infer that there exists a compact operator K on X such that

$$\lambda \notin \bigcup_{\|D\|<\varepsilon} \sigma(A+K+D).$$

So, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have $\lambda \in \rho(A+D+K)$. Therefore,

$$A + D + K - \lambda \in \Phi(X) \quad \text{and} \quad i(A + D + K - \lambda) = 0,$$

for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. It comes from [18, Theorem 7.26] that for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have

$$A + D - \lambda \in \Phi(X) \quad \text{and} \quad i(A + D - \lambda) = 0.$$

Conversely, we suppose for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have $(A + D - \lambda) \in \Phi(X)$ and $i(A + D - \lambda) = 0$. Without loss of generality, we may assume $\lambda = 0$.
Let $n = \alpha(A + D) = \beta(A + D)$, $\{x_1, \ldots, x_n\}$ be basis for $N(A + D)$ and $\{y_1^{'}, \ldots, y_n^{'}\}$ be bases for the $N((A + D)^{'})$. By [18, Theorems 2.5, 2.6] there are functionals $x_1^{'}, \ldots, x_n^{'}$ in $X^{'}$ (the adjoint space of X) and elements $y_1, \ldots, y_n$ such that

$$x_j^{'}(x_k) = \delta_{jk} \quad \text{and} \quad y_j(y_k) = \delta_{jk}, \ \ 1 \leq j, \ k \leq n, \tag{1}$$

where $\delta_{jk} = 0$ if $j \neq k$ and $\delta_{jk} = 1$ if $j = k$. The operator K is defined by :

$$Kx = \sum_{k=1}^{n} x_k^{'}(x) y_k, \ \ x \in X. \tag{2}$$

Clearly K is a linear operator defined everywhere on X. It is bounded, since

$$\|Kx\| \leq \|x\| \left(\sum_{k=1}^{n} \|x_k^{'}\| \|y_k\| \right).$$

Moreover, the range of K is contained in a finite dimensional subspace of X. Then K is a finite rank operator in X. By [18, Lemma 1.3], K is a compact operator in X. We prove that

$$N(A + D) \cap N(K) = \{0\} \quad \text{and} \quad R(A + D) \cap R(K) = \{0\}, \tag{3}$$

for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. Let $x \in N(A + D)$, then

$$x = \sum_{k=1}^{n} \alpha_k x_k,$$

therefore, $x_j(x) = \alpha_j, \ 1 \leq j \leq n$. On the other hand, if $x \in N(K)$ then $x_j^{'}(x) = 0, \ 1 \leq j \leq n$. This proves the first relation in Eq. (3). The second inclusion is similar. In fact, if $y \in R(K)$, then

$$y = \sum_{k=1}^{n} \alpha_k y_k,$$

and hence,

$$y_j(y) = \alpha_j, \ 1 \leq j \leq n.$$

But, if $y \in R(A + D)$, then,

$$y_j^{'}(y) = 0, \ 1 \leq j \leq n.$$

This gives the second relation in Eq. (3). On the other hand K is a compact operator. We deduce from [18, Theorem 7.26] that $0 \in \Phi_{A+K+D}$ and $i(A + D + K) = 0$. If $x \in N(A + D + K)$ then $(A + D)x$ is in $R(A + D) \cap R(K)$ this implies that $x \in N(A + D) \cap N(K)$ hence $x = 0$. Thus $\alpha(A + D + K) = 0$. In the same way, one proves that $R(A+D+K) = X$. Hence, $0 \in \rho(A+D+K)$. This implies that for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have $0 \notin \sigma(A + D + K)$. Also, $0 \notin \bigcap_{K \in \mathcal{K}(X)} \sigma_\varepsilon(A + K)$. So, $0 \notin \sigma_{w,\varepsilon}(A)$. □

Remark 2.4 It follows, immediately, from Theorem 2.3 and [18, Theorem 7.27] that

$$\sigma_{w,\varepsilon}(A) = \bigcup_{\|D\| < \varepsilon} \sigma_w(A + D).$$

Theorem 2.5 *Let X be a Banach space, $\varepsilon > 0$ and $A \in \mathcal{C}(X)$. Then*

$$\sigma_{w,\varepsilon}(A) := \bigcap_{F \in \mathcal{F}(X)} \sigma_\varepsilon(A + F).$$

Proof Let $\mathcal{O} := \bigcap_{F\in\mathcal{F}(X)} \sigma_\varepsilon(A+F)$. Since, $\mathcal{K}(X) \subset \mathcal{F}(X)$ we infer that $\mathcal{O} \subset \sigma_{w,\varepsilon}(A)$. Conversely, let $\lambda \notin \mathcal{O}$ then there exist $F \in \mathcal{F}(X)$ such that $\lambda \notin \sigma_\varepsilon(A+F)$. Thus, by [4, Theorem 9.2.13] we see that $\lambda \in \rho(A+D+F)$ for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. So,

$$A+D+F-\lambda \in \Phi(X) \quad \text{and} \quad i(A+D+F-\lambda)=0.$$

The use of [10, Lemma 2.1] makes us conclude that for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$,

$$A+D-\lambda \in \Phi(X) \quad \text{and} \quad i(A+D-\lambda)=0.$$

Finally, Theorem 2.3 shows that $\lambda \notin \sigma_{w,\varepsilon}(A)$. □

Remark 2.6 (i) It follows, from Theorem 2.5 that $\sigma_{w,\varepsilon}(A+F)=\sigma_{w,\varepsilon}(A)$ for all $F \in \mathcal{F}(X)$.
(ii) It is proved in [13, Section 3] that if X is a Banach space with the property DP, then $\mathcal{W}(X) \subset \mathcal{F}(X)$. Thus the Weyl pseudospectrum is invariant under weakly compact perturbations on this class of Banach spaces.

Corollary 2.7 *Let X be a Banach space and $\mathfrak{I}(X)$ be a subset of $\mathcal{L}(X)$.*
If $\mathcal{K}(X) \subset \mathfrak{I}(X) \subset \mathcal{F}(X)$, then

$$\sigma_{w,\varepsilon}(A) = \bigcap_{J\in\mathfrak{I}(X)} \sigma_\varepsilon(A+J).$$

Remark 2.8 It comes from Corollary 2.7 that $\sigma_{w,\varepsilon}(A+J)=\sigma_{w,\varepsilon}(A)$ for all $J \in \mathfrak{I}(X)$ such that $\mathcal{K}(X) \subset \mathfrak{I}(X) \subset \mathcal{F}(X)$.

Theorem 2.9 *Let $\varepsilon > 0$, A and B be two elements of $\mathcal{C}(X)$ such that $0 \notin \sigma_w(A)\cup\sigma_w(B)$. Assume that there exist two operators A_0 and $B_0 \in \mathcal{L}(X)$ such that*

$$AA_0 = I - F_1, \tag{4}$$

$$BB_0 = I - F_2, \tag{5}$$

with $F_i \in \mathcal{F}(X)$, $i=1, 2$. If the difference $A_0 - B_0 \in \mathcal{F}(X)$ then

$$\sigma_{w,\varepsilon}(A)=\sigma_{w,\varepsilon}(B).$$

Proof Using Eqs. (4) and (5) we infer that for any scalar λ

$$(A+D-\lambda)A_0 - (B+D-\lambda)B_0 = F_2 - F_1 + (D-\lambda)(A_0-B_0). \tag{6}$$

Let $\lambda \notin \sigma_{w,\varepsilon}(A)$ then $A+D-\lambda$ is Fredholm operator and $i(A+D-\lambda)=0$ for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. Since $A+D$ is closed and $\mathcal{D}(A+D)=\mathcal{D}(A)$ endowed with the graph norm is a Banach space denoted by X_{A+D} and using [17, Corollary 7.6] we obtain $\widehat{A+D}-\lambda \in \Phi^b(X_{A+D}, X)$. Moreover, $F_1 \in \mathcal{F}(X)$. Using Eq. (4) and [8, Theorem 2.1] we find that $A_0 \in \Phi^b(X, X_{A+D})$. Thus

$$(\widehat{A+D}-\lambda)A_0 \in \Phi^b(X). \tag{7}$$

Next, if the difference $A_0 - B_0 \in \mathcal{F}(X)$, applying Eq. (6) we get

$$(A+D-\lambda)A_0 - (B+D-\lambda)B_0 \in \mathcal{F}(X).$$

Also, it follows from Eq. (7) that $(\widehat{B+D}-\lambda)B_0 \in \Phi^b(X)$ and

$$i[(\widehat{B+D}-\lambda)B_0] = i[(\widehat{A+D}-\lambda)A_0] = 0. \tag{8}$$

Since $B \in \mathcal{C}(X)$, using Eq. (5) and arguing as in the last part we conclude that

$$B_0 \in \Phi^b(X, X_{B+D}).$$

Thus, since $(B+D-\lambda)B_0$ is Fredholm operator the use of [17, Theorem 7.12] shows that $\widehat{B+D}-\lambda \in \Phi^b(X_{B+D}, X)$. This implies that $B+D-\lambda$ is Fredholm operator. On the other hand, $0 \notin \sigma_e(A)\cup\sigma_e(B)$ then $i(A)=i(B)=0$. Therefore, using Eqs. (4), (5) and [8, Theorem 2.1] we have that $i(A_0)=i(B_0)=0$. This together with Eq. (6) shows that

$$i(A+D-\lambda)=i(B+D-\lambda)=0.$$

Thus, $\lambda \notin \sigma_{w,\varepsilon}(B)$. This proves that $\sigma_{w,\varepsilon}(B) \subset \sigma_{w,\varepsilon}(A)$. The opposite inclusion follows by symmetry. □

Lemma 2.10 *Let X be a Banach space, $\varepsilon > 0$, A and B two elements of $\mathcal{C}(X)$. If B is $(A+D)$-compact for all bounded operators D with $\|D\| < \varepsilon$, then*

$$\sigma_{w,\varepsilon}(A) = \sigma_{w,\varepsilon}(A+B).$$

Proof Let $\lambda \notin \sigma_{w,\varepsilon}(A)$ then for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have $A + D - \lambda$ is a Fredholm operator and $i(A + D - \lambda) = 0$. Since, B is $(A + D)$-compact, applying [18, Theorem 3.3], we get

$$\lambda \in \Phi_{A+B+D} \quad \text{and} \quad i(A + B + D - \lambda) = 0.$$

Therefore, $\lambda \notin \sigma_{e,\varepsilon}(A + B)$. We conclude that

$$\sigma_{w,\varepsilon}(A + B) \subset \sigma_{w,\varepsilon}(A).$$

Conversely, let $\lambda \notin \sigma_{w,\varepsilon}(A + B)$ then for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have $A + B + D - \lambda$ is a Fredholm operator and $i(A+B+D-\lambda) = 0$. On the other hand, B is $(A+D)$-compact, using [18, Theorem 2.12] we infer that B is $(A + B + D)$-compact, then

$$\lambda \in \Phi_{A+D} \quad \text{and} \quad i(A + D - \lambda) = 0.$$

So, $\lambda \notin \sigma_{e,\varepsilon}(A)$. This proves that

$$\sigma_{w,\varepsilon}(A) \subset \sigma_{w,\varepsilon}(A + B).$$

□

The following theorem gives a relation between the Weyl pseudospectrum of the sum of the two bounded linear operators and the Weyl pseudospectrum of each of these operators.

Theorem 2.11 *Let X be a Banach space, $\varepsilon > 0$, A and B two elements of $\mathcal{L}(X)$.*
If for all bounded operators D such that $\|D\| < \varepsilon$ we have $A(B + D) \in \mathcal{F}(X)$, then

$$\sigma_{w,\varepsilon}(A + B)\backslash\{0\} \subseteq \big[\sigma_w(A) \cup \sigma_{w,\varepsilon}(B)\big]\backslash\{0\}.$$

If further $(B + D)A \in \mathcal{F}(X)$, then

$$\sigma_{w,\varepsilon}(A + B)\backslash\{0\} = \big[\sigma_w(A) \cup \sigma_{w,\varepsilon}(B)\big]\backslash\{0\}.$$

Proof For $\lambda \in \mathbb{C}$, we can write

$$(\lambda - A)(\lambda - B - D) = A(B + D) + \lambda(\lambda - A - B - D) \tag{9}$$

and

$$(\lambda - B - D)(\lambda - A) = (B + D)A + \lambda(\lambda - A - B - D). \tag{10}$$

Let $\lambda \notin \big[\sigma_w(A) \cup \sigma_{w,\varepsilon}(B)\big]\backslash\{0\}$. Then, $(\lambda - A) \in \Phi(X)$ and for all $\|D\| < \varepsilon$, $(\lambda - B - D) \in \Phi(X)$. It follows from [17, Theorem 5.7] that

$$(\lambda - A)(\lambda - B - D) \in \Phi(X).$$

Since $A(B + D) \in \mathcal{F}(X)$, applying Eq. (9), we have $(\lambda - A - B - D) \in \Phi(X)$ then $\lambda \notin \sigma_{e,\varepsilon}(A + B)$. Therefore

$$\sigma_{w,\varepsilon}(A + B)\backslash\{0\} \subseteq \big[\sigma_w(A) \cup \sigma_{w,\varepsilon}(B)\big]\backslash\{0\}. \tag{11}$$

Now, we prove the inverse inclusion of Eq. (11).
Suppose $\lambda \notin \sigma_{e,\varepsilon}(A+B)\backslash\{0\}$, then for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have $(\lambda - A - B - D) \in \Phi(X)$. Since $A(B + D) \in \mathcal{F}(X)$, $(B + D)A \in \mathcal{F}(X)$ then by Eqs. (9) and (10), we have

$$(\lambda - A)(\lambda - B - D) \in \Phi(X) \text{ and } (\lambda - B - D)(\lambda - A) \in \Phi(X).$$

Applying [15, Theorem 6], it is clear that $(\lambda - A) \in \Phi(X)$ and for all $\|D\| < \varepsilon$ we have $(\lambda - B - D) \in \Phi(X)$. Therefore $\lambda \notin \sigma_w(A) \cup \sigma_{w,\varepsilon}(B)$. This proves that

$$\sigma_{w,\varepsilon}(A + B)\backslash\{0\} = \big[\sigma_w(A) \cup \sigma_{w,\varepsilon}(B)\big]\backslash\{0\}.$$

□

References

1. Abdmouleh, F.; Ammar, A.; Jeribi, A.: Pseudo-Browder essential spectra of linear operators and application to a transport equations. Preprint (2012)
2. Browder, F.: On the spectral theory of elliptic differential operators. I. Math. Ann. **142**, 22–130 (1961)
3. Davies, E.B.: Spectral Theory and Differential Operators. Cambridge Studies in Advanced Mathematics, 42. Camb. Univ. Press, Cambridge (1996)
4. Davies, E.B.: Linear Operators and their Spectra. Cambridge Studies in Advanced Mathematics, 106. Camb. Univ. Press, Cambridge (2007)
5. Dautray, R.; Lions, J.L.: Analyse mathématique et calcul numérique. Tome 9, Masson, Paris, (1988)
6. Harte, R.: Fredholm, Weyl and Browder theory. Proc. Roy. Irish Acad. Sect. A 85, no. 2, 151–176 (1985)
7. Hinrichsen, D.; Pritchard, A.J.: Robust stability of linear operators on Banach spaces. J. Cont. Opt. **32**, 1503–1541 (1994)
8. Jeribi, A.: Fredholm operators and essential spectra. Arch. Ineq. Appl. 2, no. **2–3**, 123–140 (2004)
9. Jeribi, A.; Mnif, M.: Fredholm operators, essential spectra and application to transport equation. Acta Appl. Math. **89**, 155–176 (2006)
10. Jeribi, A.; Moalla, N.: A characterization of some subsets of Schechter's essential spectrum and application to singular transport equation. J. Math. Anal. Appl. **358**, 434–444 (2009)
11. Kato, T.: Perturbation theory for nullity, deficiency and other quantities of linear operators. J. Anal. Math. **6**, 261–322 (1958)
12. Landau, H.J.: On Szegös eingenvalue distribution theorem and non-Hermitian kernels. J. Anal. Math. **28**, 335–357 (1975)
13. Latrach, K.: Essential spectra on spaces with the Dunford-Pettis property. J. Math. Anal. Appl. **223**, 607–623 (1999)
14. Lutgen, J.: On essential spectra of operator-matrices and their Feshbach maps. J. Math. Anal. Appl. **289**, 419–430 (2004)
15. Müller, V.: Spectral theory of linear operators and spectral system in Banach algebras. Oper. Theo. Adva. Appl. **139** (2003)
16. Nussbaum, R.D.: Spectral mapping theorems and perturbation theorems for Browder's essential spectrum. Trans. Amer. Math. Soc. **150**, 445–455 (1970)
17. Schechter, M.: Principles of functional analysis. Grad. Stud. Math. vol. 36, Amer. Math. Soc., Providence, RI (2002)
18. Schechter, M.: Spectra of Partial Differential Operators. North-Holland, Amsterdam, New York (1986)
19. Trefethen, L.N.: Pseudospectra of Matrices, Numerical analysis 1991 (Dundee, 1991), 234–266. Pitman Res. Notes Math. Ser., **260**, Longman Sci. Tech., Harlow (1992)
20. Trefethen, L.N.; Embree, M.: Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators. Prin. Univ. Press, Princeton and Oxford (2005)
21. Varah, J.M.: The Computation of Bounds for the Invariant Subspaces of a General Matrix Operator. Stan. Univ. Comp. Sci., Dept. Tech. Report (1967)